Bifurcation picture for gap solitons in nonlinear modulated systems

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We investigate the problem of nonlinear wave propagation in periodic media. Four different classes of periodic nonlinear media are taken into consideration: a nonlinear diatomic elastic chain, modulated nonlinear optical media, a diatomic easy-axis ferromagnetic chain, and an easy-plane antiferromagnet in an external magnetic field. The main result of our work is a qualitative analysis of all kinds of small amplitude soliton excitations with frequencies lying in the gap and near the gap of the linear wave spectrum. We also study the evolution of the system phase portrait and the bifurcation picture of the soliton solutions under changes of the medium parameters. [S1063-651X(99)06408-9]

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I. INTRODUCTION

Generally the soliton dynamics of nonlinear systems have been investigated essentially in the framework of simple models of the homogeneous medium. Nowadays a new and interesting problem is the propagation of a nonlinear wave in a periodic medium. The periodicity of the structure leads to the initiation of the gap or gaps (stop bands) in the dispersion law of linear excitations, and the existence of such a gap affects essentially the structure and properties of solitons with the parameters lying near the gap. It is well known that the condition of existence of two-parameter solitons for a fixed sign of the nonlinearity is related to the sign of a linear wave dispersion. In the case of the existence of a periodical nonlinear medium near the gap of the spectrum, there are two branches of the spectrum with opposite dispersion signs. The width of a gap with a low-amplitude modulation of the medium properties is small, and excitations from these two branches interact strongly between themselves. Thus the answer to the question concerning the character of combined two-component solitons is not evident in this situation. In a modulated medium some physical parameters vary periodically with the coordinate. The nonlinear optical medium with the modulation of the refractive index gives us the example of a modulated system [1–3]. The one-dimensional unharmonic diatomic chain represents another example [4].

One-dimensional unharmonic elastic and magnetically ordered chains have become a classical object of investigation of nonlinear and soliton dynamics [5,6]. The simplest and most natural generalization of a homogeneous chain is a diatomic chain with periodically arranged atoms of two different types. In a preceding paper [4] we gave the simplest example of a nonlinear elastic diatomic chain with two different masses and with nearest-neighbor interaction. Some properties of the gap soliton in such a chain differ from those of the Bragg solitons in modulated optical media due to the discreteness of the nonlinear chain. In this paper we will consider diatomic elastic and magnetic chains with more general types of modulation [7–9].

II. FORMULATION OF THE PROBLEM AND THE MAIN DYNAMICAL EQUATIONS

(i) Primarily we consider a one-dimensional periodic diatomic chain with atoms of masses M and m $(M>m)$ and an unharmonic potential of the nearest-neighbor interaction and on-site potential. We choose the even interparticle potential

$$U(\xi_n)=\frac{A}{2}\xi_n^2+\frac{C}{4}\xi_n^4, \tag{1}$$

and even on-site potential

$$V(u_n)=\frac{\alpha}{2}u_n^2+\frac{\beta}{4}u_n^4, \tag{2}$$

where u_n is the nth atom displacement, $\xi_n=u_n-u_{n-1}$, and we take the constants A and α to be positive.

The corresponding equation of motion for the nth particle has the form

$$\left\{M\frac{d^2u_n}{dt^2}+A(2u_n-u_{n+1}-u_{n-1})+\alpha u_n\right\} + \left\{(m-M)\frac{d^2u_n}{dt^2}\;\delta(n-2s)\right\}+\{C(u_n-u_{n+1})^3 + C(u_n-u_{n-1})^3+\beta u_n^3\}=0, \tag{3}$$

where in brackets we have separated the linear (L), modulated (M), and nonlinear (N) parts of the equation.

(ii) For comparison, the corresponding equation for nonlinear electromagnetic waves in periodic medium is [1]

$$\left\{\frac{\partial^2E}{\partial x^2}-\frac{\partial^2E}{\partial t^2}\right\}-\left\{\delta\cos\left(\frac{x}{l}\right)\frac{\partial^2E}{\partial t^2}\right\}-\left\{\gamma|E|^2\;\frac{\partial^2E}{\partial t^2}\right\}=0. \tag{4}$$

In addition, we consider two simple models of magnetically ordered modulated chains.

(iii) The dynamics of the easy-axis ferromagnet with two magnetic sublattices with different lengths of the spins is expressed by the equations

$$\frac{d\vec{M}_n}{dt}+J[\vec{M}_n\vec{m}_n]+J[\vec{M}_n\vec{m}_{n-1}]+\beta[\vec{M}_n\vec{e}](\vec{M}_n\vec{e})=0,$$

$$n=2s, \tag{5}$$

$$\frac{d\vec{m}_n}{dt}+J[\vec{m}_n\vec{M}_n]+J[\vec{m}_n\vec{M}_{n+1}]+\beta[\vec{m}_n\vec{e}](\vec{m}_n\vec{e})=0,$$

$$n=2s+1, \tag{6}$$

where $\vec{M}_n$ and $\vec{m}_n$ are the magnetizations of the sublattices, J is the exchange integral, and β is an anisotropy constant. For examples of such systems see, e.g., Ref. [10].

(iv) The dynamical equations for the biaxial antiferromagnet with a strong easy-plane anisotropy in the presence of an in-plane magnetic field has the form

$$\left(\frac{1}{\omega_0^2}\right)\frac{d^2\Phi_n}{dt^2}-\frac{J}{\beta}[\sin(\Phi_n-\Phi_{n+1})+\sin(\Phi_n-\Phi_{n-1})]$$

$$+h\sin(\Phi_n)+\frac{\gamma}{\beta}\sin(\Phi_n)\cos(\Phi_n)=0, \tag{7}$$

where β is the constant of strong easy-plane anisotropy, γ is the constant of small in-plane anisotropy, $h=H/\beta M_0$, $\omega_0=2\mu_0 M_0\beta/h$, M_0 is the equilibrium magnetization of the sublattice, and μ_0 is the Bohr magneton. The variables Φ_n denote the angles in an easy plane between the spin direction and the direction of easy axis. (The external field is parallel to this axis.)

III. DISPERSION LAW OF LINEAR WAVES

In the above-listed cases the dispersion law of linear waves (phonons, photons, or magnons) has a gap at $k_0=\pi/2a$, where k is a wave number and a is the lattice spacing. [In the case of Eq. (4) the wave number of the gap k is $k_0=\pi/2l$.] The most interesting part of the spectrum is in the vicinity of k_0 (at $k=k_0+\kappa$, with $\kappa a\ll 1$), where two branches of the spectrum have the standard form. For example, in the case of an elastic diatomic chain the lower branch (I) near the gap corresponds to the oscillations of heavy atoms with near-opposite phases. This branch ends at the point $\omega_1^2=(2A+\alpha)/M$. The upper boundary of the gap is at frequency $\omega_2^2=(2A+\alpha)/m$, and for this branch (u) the heavy particles essentially do not move, while the light atoms vibrate with opposite phases. The displacements of the particles can be presented in the form

$$u_n=v_n(t)\sin\left(\frac{\pi n}{2}\right)+w_n(t)\cos\left(\frac{\pi n}{2}\right), \tag{8}$$

where the functions w and v slowly depend on the coordinate. In the long-wave approximation we can change the discrete argument n by the continuum coordinate x/a, and reduce Eq. (3) to the following system of partial differential equations for the functions $w(x,t)$ and $v(x,t)$:

$$M\frac{\partial^2 v}{\partial t^2}+(2A+\alpha)v+2A\alpha\frac{\partial w}{\partial x}=0, \tag{9}$$

$$m\frac{\partial^2 w}{\partial t^2}+(2A+\alpha)w-2A\alpha\frac{\partial v}{\partial x}=0. \tag{10}$$

The essential feature of this set of equations is that they contain only the first space derivatives. Such a property of these dynamical equations is specific to the value of the wave number $k_0=\pi/2a$ and the small width of the gap $\Delta=\omega_2^2-\omega_1^2$ [$\Delta=(2A+\alpha)(M-m)/Mm$ for the diatomic chain]. This fact gives the possibility of a qualitative analysis of a dynamical system on the phase plane. Linear waves of the forms $v\sim\sin(\kappa x)\sin(\omega t)$ and $w\sim\cos(\kappa x)\sin(\omega t)$ have the dispersion law

$$\omega_{(u,l)}^2=\omega_{2,1}^2\pm\frac{4A^2}{(2A+\alpha)(M-m)}(a\kappa)^2 \tag{11}$$

for upper (u) and lower (l) branches of the spectrum. [The dropped second space derivatives in Eqs. (9) and (10) give rise to the small terms in Eq. (11) of order $(a\kappa)^2$, which do not contain the large parameter $(M-m)^{-1}$.]

The approximate solution of Eq. (4) may be written as $E=v(x,t)\sin(x/2l)+w(x,t)\cos(x/2l)$, and the dispersion law for $\delta\ll 1$ is

$$\omega_{(u,l)}^2=\omega_{2,1}^2\pm\frac{2\kappa^2}{\delta}, \tag{12}$$

where $\omega_{2,1}^2=(1\pm\delta/2)/4l^2$ and $\Delta=\delta/4l^2$.

A similar method also gives the dispersion law with the two zones for the ferromagnetic diatomic chain [Eqs. (5) and (6)]. The upper branch $\omega_{(u)}(\kappa)$ corresponds to the rotation of the ''long'' spin with length M_0, and the lower branch to that of the ''short'' spin m_0. We have the dependence $\omega(\kappa)$,

$$\omega_{(u,l)}\cong\omega_{2,1}\pm\frac{2M_0 m_0 J^2}{(2J-\beta)(M_0-m_0)}(\kappa a)^2, \tag{13}$$

with $\omega_2=2JM_0+\beta m_0$, $\omega_1=2Jm_0+\beta M_0$, and $\Delta=\omega_2-\omega_1=(2J-\beta)(M_0-m_0)$.

Finally, for a one-dimensional antiferromagnet in a magnetic field [see Eq. (7)] in the collinear phase (with $\Phi_n=\pi n$ in ground state), the dispersion law has the form

$$\left(\frac{\omega^2}{\omega_0}\right)=\frac{2J}{\beta}+\frac{\gamma}{\beta}\pm h\pm\frac{2J}{\beta^2 h}(\kappa a)^2. \tag{14}$$

The dependencies (11)–(14) have the same form

$$\omega_{(u,l)}\cong\omega_{2,1}\left\{1\pm C\frac{(\kappa a)^2}{\nu^2}\right\},\quad \Delta\sim\nu^2, \tag{15}$$

where the small parameter ν gives the amplitude of the modulation of periodic media $(M-m,\delta,M_0-m_0,h\sim\nu^2)$ and $C=\text{const}$. The dispersion $D=\partial^2\omega/\partial x^2$ of two branches of the spectrum has opposite signs.

IV. QUALITATIVE ANALYSES OF SOLITONS NEAR THE GAP OF THE SPECTRUM

Let us bring the nonlinearity into consideration. The structures of nonlinear waves and solitons depend essentially on the character of a linear wave dispersion and the type of nonlinearity. The simplest situation appears in the case of cubic unharmonicity in the dynamical equations (as in all above-listed models), which corresponds to the natural interactions of elementary excitations of the ''density-density'' type. We can introduce a ''nonlinear dispersion law'' for the spatially homogeneous nonlinear waves $\omega=\omega(\kappa,u)$, where ω, κ, and u are the frequency, wave number, and amplitude of the nonlinear wave. The sign of the derivative $\partial^2\omega/\partial u^2$ characterizes the nonlinearity of the system. In the case of ''hard'' (''soft'') nonlinearity, we have $\partial^2\omega/\partial u^2>0$ ($\partial^2\omega/\partial u^2<0$). In the first example (i), the nonlinearity is ''hard,'' and in all other [(ii), (iii), and (iv)] it is ''soft.'' But due to the symmetry of the spectrum near the gap, the change of the nonlinearity sign does not give new physical results. It is well known that the homogeneous nonlinear waves are modulationally unstable under the condition $(\partial^2\omega/\partial k^2)(\partial^2\omega/\partial u^2)<0$, and such an instability leads to envelope soliton creation. When the sign in the criterion is changed, homogeneous nonlinear waves become stable and the existence of specific ''dark'' solitons becomes possible. As the criterion has opposite signs for two branches of the spectrum near the gap, the envelope solitons in this range of frequencies must represent the complex combination of dark and bright solitons.

We consider the nonlinear diatomic elastic chain (i) at first. Let us introduce normalized displacements of the particles and normalized coordinate:

$$W_n=w_n\sqrt{\frac{6|C|}{4(2A+\alpha)}},\quad V_n=v_n\sqrt{\frac{6|C|}{4(2A+\alpha)}},\tag{16}$$

$$z=\frac{(2A+\alpha)}{2A}\,\frac{x}{\alpha}.\tag{17}$$

Substitution of relation (8) into Eq. (3) gives the set of equations in the long-wave approximation,

$$\frac{V_{tt}}{\omega_1^2}+W_z+V+\frac{4}{3}(\xi V^3+3\sigma VW^2)=0,\tag{18}$$

$$\frac{W_{tt}}{\omega_2^2}-V_z+W+\frac{4}{3}(\xi W^3+3\sigma WV^2)=0,\tag{19}$$

where $\xi=1+\beta/2C$, $\sigma=\mathrm{sgn}\ C$, $V_{tt}=\partial^2V/\partial t^2$, and $V_z=\partial V/\partial z$.

Below we consider the stationary (periodical in time) small-amplitude solutions of Eqs. (18) and (19) using the method of asymptotical expansion. For the small parameter of expansion it is convenient to choose the value $\epsilon^2=(\omega^2-\omega_1^2)/\omega_1^2$. In the case of a small amplitude of the modulation of the parameters of the medium, the parameter ϵ remains small everywhere inside and near the gap of the linear spectrum. To the lowest-order ϵ (resonance) approximation in which

$$W\cong-f\sin(\omega t),\quad V\cong g\sin(\omega t),\tag{20}$$

we obtain the system of ordinary differential equations

$$f_z=g[(1-\Omega^2)+(\xi g^2+3f^2)],\tag{21}$$

$$-g_z=f[(1+\nu^2-\Omega^2)+(\xi f^2+3g^2)],\tag{22}$$

where we have introduced the normalized frequency $\Omega=\omega/\omega_1$, the width of the gap $\Delta=\nu^2=(M-m)/m$, and the main approximation $\omega_1=\omega_2$.

For the optical model (ii) the corresponding substitution in the main approximation has a form

$$E\cong\sqrt{\frac{4}{3}}\gamma\sin(\omega t)\ \left\{f(x)\sin\left(\frac{x}{2l}\right)+g(x)\cos\left(\frac{x}{2l}\right)\right\},\tag{23}$$

and in this case Eq. (4) is reduced to a system of two equations for the functions f and g:

$$f_z=g[(1-\Omega^2)-(g^2+f^2)],\tag{24}$$

$$-g_z=f[(1+\nu^2-\Omega^2)-(f^2+g^2)],\tag{25}$$

where $\Omega=2l\omega=\omega/\omega_1$, $\nu^2=\delta/2$, and $z=x/2l$.

In the diatomic ferromagnetic chain (iii) for nonlinear magnons with wave numbers lying near the middle of the Brillouin zone, it is reasonable to represent the magnetization vectors $\vec{M}_n$ and $\vec{m}_n$ in the forms

$$M_n^x+iM_n^y\to\sqrt{2}M_0(-1)^n\left\{1+\frac{\beta M_0}{2Jm_0}\right\}^{1/2}\exp(i\omega t)g(x),\tag{26}$$

$$m_n^x+im_n^y\to\sqrt{2}m_0(-1)^n\left\{1+\frac{\beta M_0}{2Jm_0}\right\}^{1/2}\exp(i\omega t)f(x).$$

Then Eqs. (5) and (6) are reduced to the following system of equations:

$$f_z=g[(1-\Omega)-(\zeta g^2+f^2)],\tag{27}$$

$$-g_z=f\left[\frac{m_0}{M_0}(1+\nu-\Omega)-\left(\zeta\left(\frac{m_0}{M_0}\right)^2f^2+g^2\right)\right],\tag{28}$$

where $\Omega=\omega/\omega_1$, $z=2x(1+\zeta)/a$, $\zeta=\beta M_0/2Jm_0$, and $\nu=\omega/\omega_1-1$.

Finally, in case (iv) of an easy-plane antiferromagnet in a magnetic field, it is useful to introduce the slowly varying variables $f(x)$ and $g(x)$:

$$\Phi_n=\pi n+2\sqrt{\frac{J}{\beta}}\frac{\omega_0}{\omega_1}\sin(\omega t)\left\{g\sin\left(\frac{\pi n}{2}\right)-f\ \cos\left(\frac{\pi n}{2}\right)\right\}.\tag{29}$$

Then the system of equations has the forms

$$f_z=g[(1-\Omega^2)-(\eta_-g^2+3f^2)],\tag{30}$$

$$-g_z=f[(1+\nu^2-\Omega^2)-(\eta_+f^2+3g^2)],\tag{31}$$

where $\Omega=\omega/\omega_1$, $\eta_\pm=1+2\gamma/J\pm\beta h/2J$, $\nu^2=2h\beta/(2J+\gamma-h\beta)$, and $z=x\beta(\omega_1/\omega_0)^2/2J$.

In all above-listed examples [(i)–(iv)] the equations for the slowly varying amplitudes of two components of the field (f and g) have the same form, and are reduced to the following systems of general type:

$$f_z=g[\lambda(\omega_1-\omega)+(pg^2+\sigma f^2)], \tag{32}$$

$$-g_z=f[\mu(\omega_2-\omega)+(qf^2+\sigma g^2)], \tag{33}$$

where $|\sigma|=1$. The sign of σ characterizes the type of interaction between excitations of the different branches of the spectrum, and the signs of p and q characterize the interaction between excitations of the same branch. The excitations are attracted together under condition $p,q,\sigma<0$, and are repelled from one another while $p,q,\sigma>0$. Without loss of generality we can assume that $\lambda,\mu>0$. Then the changing of the signs of p, q, and σ leads only to a changing of the direction of the modification of the bifurcation picture with the frequency (see below).

Equations (32) and (33) represent the Hamiltonian equations for the dynamical Hamiltonian system with one degree of freedom, and with the following Hamiltonian:

$$H=\lambda(\omega_1-\omega)\frac{g^2}{2}+\mu(\omega_2-\omega)\frac{f^2}{2}+p\frac{g^4}{4}+q\frac{f^4}{4}+\sigma\frac{g^2f^2}{2}. \tag{34}$$

The variables f and g play the roles of the generalized canonically conjugate coordinate and momentum.

We made some assumptions and simplifications to obtain the effective "dynamical" system of equations (32) and (33) with one degree of freedom from the initial discrete dynamical equations (3)–(7). An investigation of soliton dynamics in the framework of this simple system describes the main features of the nonlinear dynamic of the initial system, though the initial one is much more complicated. For example, it is well known that the nonlinear dynamics of discrete systems, even in the case of a small number of degrees of freedom, can be chaotic. It occurs at so-called bifurcation points, i.e., when the system or solution parameters are close to values which correspond to the essential solution transformation. But numerical calculations show that the dynamics of the system become regular out of the vicinity of these points. Subsequently solitonlike localized excitations are stable in these regions (see, for example, Ref. [12]). So we hope that solutions for gap and near-gap solitons, which we shall obtain using the effective equations (32) and (33), will be stable in domains of parameters far from the bifurcation points. About the stability of gap solitons, see Ref. [13]. The chaotic dynamics of gap solitons and their stability are the questions of special interest which we shall not discuss in the paper.

The existence of the integral of motion H allows one to integrate the system of equations (32) and (33) exactly. The solutions for solitons of different kinds can be easily expressed in terms of the hyperbolic functions [4], but it is useful to use methods of qualitative analysis of dynamical systems and consider possible solutions of the system of equations (32) and (33) in the phase plane (g,f). Attention should be paid to separatricies, which correspond to soliton solutions. The phase portrait of the system depends on the signs and on the value of the parameters λ, μ, p, q, and σ, and the value of the frequency ω.

V. GAP SOLITONS IN A DIATOMIC ELASTIC CHAIN

At first we paid special attention to the soliton dynamics of a diatomic elastic chain with a "hard" nonlinearity of the on-site potential, i.e., on Eqs. (32) and (33) with $\lambda=\mu=1$, $p=q=(1+\beta/2C)/3$, and $\sigma=+1$ $(C>0)$. In this case the phase portrait of the system on the (g,f) plane depends on the relation between the frequency ω and the medium parameter p. The dynamical solutions transform essentially, and new solutions appear by way of bifurcation on some lines $\omega_i=\omega_i(p)$ on the plane of parameters ω,p. The general theory of bifurcations is given, e.g., in the book by Arnold [11].

It is convenient for us to discuss the bifurcation picture of soliton solutions in terms of the dependence of the fixed points on the frequency ω and parameter p.

There are four domains of the values of parameter p: (a) $p>1$ $(\beta>4C)$, (b) $0<p<1$ $(2C<\beta<4C)$, (c) $-1<p<0$ $(-8C<\beta<2C)$, and (d) $p<-1$ $(\beta<-8C)$. In the first region (a) for $\omega<\omega_1$, the single fixed point in the phase plane is the center at $g=f=0$, and the separatricies and subsequently soliton solutions are absent. At $\omega=\omega_1$ the first bifurcation occurs: the center is split, and for the frequency range in $\omega_1<\omega<\omega_2$ the system possesses a saddle point at $g=f=0$ and two centers at points $f=0, g=\pm\sqrt{(\omega-\omega_1)/p}$. In such a frequency gap there are two separatrix loops, which issue out of the saddle, envelop the center, and enter into the same saddle. At $\omega=\omega_2$ the second bifurcation occurs: the singular point $g=f=0$ (saddle) splits into the center at the point $(f=0,g=0)$ and two new saddle points $[g=0$ and $f=\pm\sqrt{(\omega-\omega_2)/p}]$. Now these two saddles are bound together by two different separatrix loops, which envelop the center $g=f=0$ and the center $f=0$, $g=\pm\sqrt{(\omega-\omega_1)/p}$, and correspond to the two different types of soliton solutions. Finally, at frequency $\omega=\omega_*=(pw_2-w_1)/(p-1)$, at which the last bifurcation occurs, each of the center points $[f=0,\ g=\pm\sqrt{(\omega-\omega_1)/p}]$ splits into the saddle points $f=0,g=\pm\sqrt{(\omega-\omega_1)/p}$ and new centers in the points $f_0=\pm\sqrt{(\omega-\omega_*)/(p+1)}, g_0=\pm\sqrt{f_0^2+(\omega_2-\omega_1)/(p-1)}$. The discussed bifurcation picture for the dependencies of the singular points g and f on the frequency is shown schematically in Fig. 1(a). (The solid lines correspond to the centers and the dashed lines correspond to the saddles.)

The phase portraits for different frequencies are shown in Fig. 2 (see domain A), where the horizontal axis in all phase pictures correspond to the f field, and the vertical axis to the g field. The profiles of the g and f fields in all solitons can be easily obtained from the phase portaits. It is clear from Eqs. (32) and (33) that in the linear limit the lower boundary of the gap $(\omega=\omega_1)$ corresponds to the excitations with $g\neq0$ and $f=0$, i.e., the g field describes the excitations from the lower branch of the spectrum. In contrast, the upper branch corresponds to the excitations of the f field, and on the upper boundary of the gap $(\omega=\omega_2)$ we have $g=0$ and $f\neq0$. In the case of the diatomic elastic chain, the g field describes the opposite-phase vibrations of heavy atoms with mass M. Let

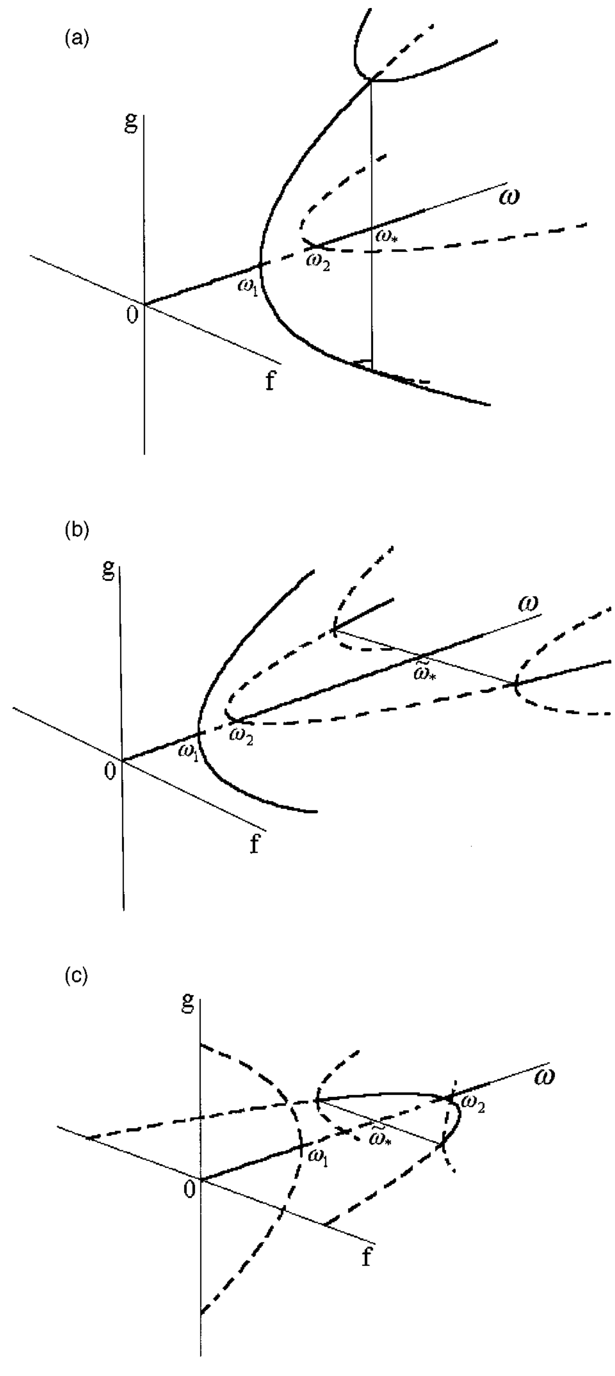

FIG. 1. (a) Bifurcation picture for the diatomic elastic chain in the case of $p>1$. The solid lines correspond to the centers, and the dashed lines to the saddles. (b) Bifurcation picture for the diatomic elastic chain in the case of $0<p<1$. The solid lines correspond to the centers, and the dashed lines to the saddles. (c) Bifurcation picture for the diatomic elastic chain in the case of $-1<p<0$. The solid lines correspond to the centers, and the dashed lines to the saddles. (d) Bifurcation picture for the diatomic elastic chain in the case of $p<-1$. The solid lines correspond to the centers, and the dashed lines to the saddles. (e) Bifurcation picture for the modulated nonlinear optical media. The solid lines correspond to the centers, and the dashed lines to the saddles. (f) Bifurcation picture for the modulated ferromagnetic. The solid lines correspond to the centers, and the dashed lines to the saddles.

us note that in the soliton for $\omega_1<\omega<\omega_2$ only the curve for the field $g(x)$ has the standard soliton form. The opposite-phase vibrations of the light particles (the f field) are essentially smaller. The soliton can be considered as bound vibrations of the lower branch, which are accompanied by localized light-atom vibrations, i.e., the soliton of the lower branch localized phonons of the upper branch. We will call this type of soliton an S soliton. In the region $\omega_2<\omega<\omega_*$, solitons of two types (S and C) exist. The S soliton is similar to the S soliton in the gap, but is accompanied by vibrations of light atoms (f field) with a finite amplitude at infinity ($x\rightarrow\pm\infty$). It is important that solitons uniform in space for the f field are stable while $\omega>\omega_2$, because the dispersion $\partial^2\omega/\partial k^2$ for the upper branch is positive. That is why we can study the solitons of the f field with nonvanishing asymptotes in this region of ω. The C-type soliton has a

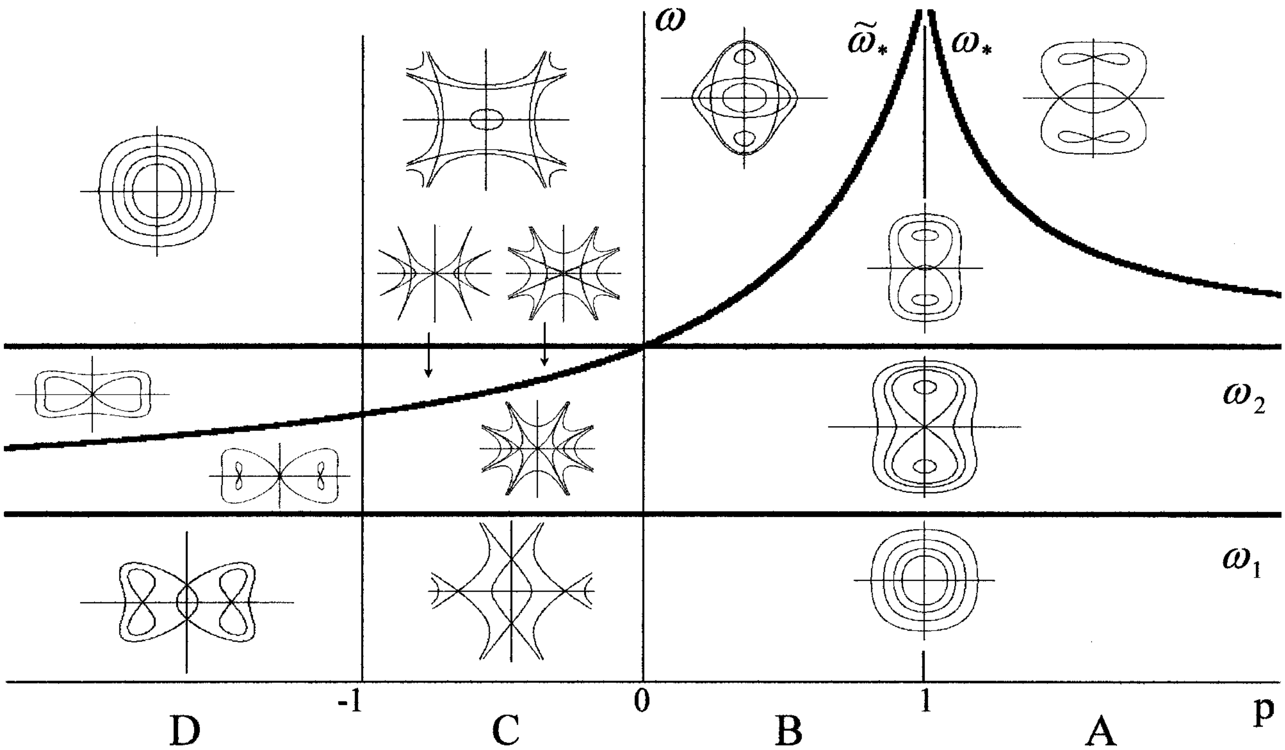

FIG. 2. The general evolution of phase portraits of the system of equations (32) and (33) with $\lambda=\mu$, $p=q$, and $\sigma=1$.

different form. The amplitude of the f field is essentially larger than that for the g field and, in this sense, this is the soliton of the upper branch. The envelope for light atom vibrations has the form of a kink in this case. This result is in complete agreement with the analysis of the dynamics of a dark soliton in a monoatomic unharmonic chain. Finally, at frequency $\omega>\omega_*$ in addition to solitons of C and S type, there exists a new type (K) of soliton in which the localized excitation of the f field exists against the background of a nonvanishing g field. It is easy to see that this soliton is a combination of the bright localized soliton of the f field with the dark soliton on the pedestal of the g field, i.e., the excitations of the f field are localized in the ''hole'' in condensate of the excitations of the g field.

In the limit $M=m$ we have a monoatomic chain with $\omega_1=\omega_2=\omega_*$ for $k=\pi/2a$ and without a gap. The phase portrait differs essentially from that given above in this case [see Fig. 3(a)]. There are no solitons now, but only specific kinks of two types. In such a ''phase kink'' the g field tends to zero at one infinity ($x\rightarrow\infty$), and the f field tends to zero at another infinity ($x\rightarrow-\infty$). The fields f and g are identical to one another as in the limit $M=m$, and describe the opposite-phase vibrations of the odd and even particles; the ''phase kink'' is the localized phase shift in a nonlinear homogeneous standing wave. The value of this shift is equal to the lattice spacing a. In the diatomic chain, a K soliton represents the bound state of two different ''phase kinks'' with opposite signs.

Let us study the bifurcation picture in the second region (b) of parameter p ($0<p<1$ or $2C<b<4C$). In this case, like the above picture, three bifurcations in the phase plane take place when the frequency ω grows. The bifurcations at $\omega=\omega_1$ and $\omega=\omega_2$ have the same character as in the above occasion, and solitons with frequencies lying in the gap of the spectrum and near the gap have the same form as for $p>1$. But the third bifurcation at $\omega=\tilde{\omega}_*=(\omega_2-p\omega_1)/(1-p)$ differs significantly from the bifurcation at $\omega=\omega_*$ for $p>1$. Now at $\omega=\tilde{\omega}_*$ the saddle points $g=0$ and $f=\pm\sqrt{(\omega-\omega_2)/p}$ split into a center at point $g=0$, $f=\pm\sqrt{(\omega-\omega_2)/p}$ and two saddle points $g_0=\pm\sqrt{(\omega-\tilde{\omega}_*)/(1+p)}$ and $f_0=\pm\sqrt{g_0^2+(\omega_2-\omega_1)/(1-p)}$ [Fig. 1(b)]. An analysis shows that for $\omega>\tilde{\omega}_*$ four different types of separatrices S, C, $\tilde{S}$, and $\tilde{C}$ coexist. Separatrices of S and C type are continuously derived from ones considered for the case $\omega<\tilde{\omega}_*$ and the separatrices $\tilde{S}$ and $\tilde{C}$ correspond to new solitons. The main feature of the latter appears to be the fact that now none of the field vanishes at infinity, and the main field difference between S and C solitons disappears — they differ only qualitatively by the amplitude of the field at the soliton center. Thus, a symmetry between g and f fields arises in $S(C)$ and $\tilde{S}(\tilde{C})$ solitons. In the special case of $p=1$ the third bifurcation point goes to infinity ($\tilde{\omega}_*\rightarrow\infty$), and we have a simple bifurcation picture similar to those for a nonlinear modulated optical medium investigated in Refs. [1,3] [in the case of hard nonlinearity with $\gamma<0$ in Eq. (4)].

In the limit of a monoatomic elastic chain with $M=m$, solitons of C, S, $\tilde{C}$, and $\tilde{S}$ type transform into ''phase kinks'' as in the above case $p>1$, but now the structure of these kinks is somewhat different from that of the above kinks. Now the phase kink describes the phase shift by the interatomic spacing a in a standing nonlinear wave of finite amplitude, and separates two domains with the following structure of the atomic oscillations: $(\ldots{}_{\uparrow\uparrow}{}^{\downarrow\downarrow}{}_{\uparrow\uparrow}{}^{\downarrow\downarrow}\ldots)$. [In the above case $p>1$, the structure of the atomic oscillations has the form $(\ldots{}_{\uparrow}0^{\downarrow}0_{\uparrow}0^{\downarrow}0_{\uparrow}\ldots)$, where indexes (0) denote nonmoving particles.] The bifurcation picture and phase por-

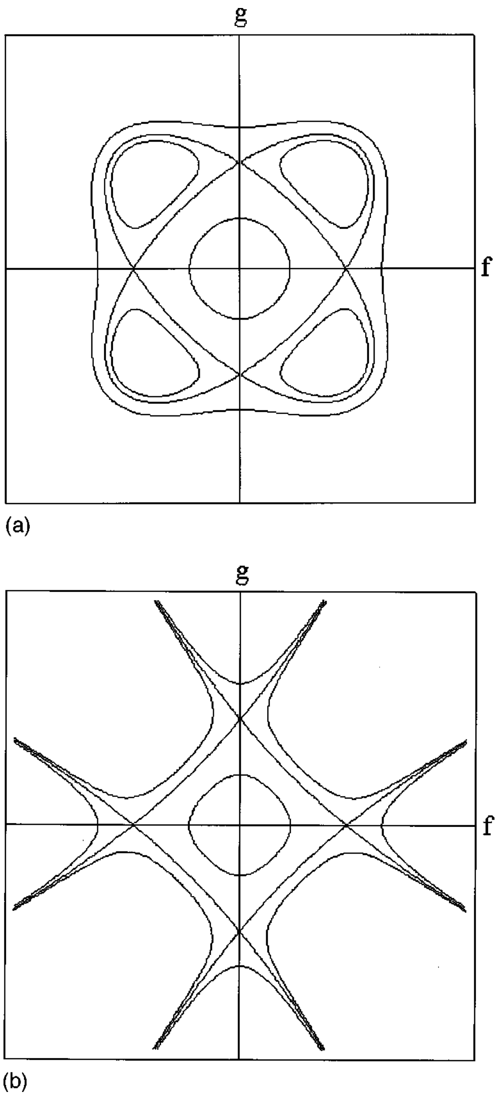

FIG. 3. (a) Phase portrait of the system of equations (32) and (33) in the limit of $M=m$, with $\lambda=\mu$, $p=q>1$, $\sigma=1$, and $\omega_1<\omega$. (b) Phase portrait of the system of equations (32) and (33) in the limit of $M=m$, with $\lambda=\mu$, $p=q$, $-1<p<0$, $\sigma=1$, and $\omega<\omega_1$.

traits for case (b) are presented in Figs. 1(b) and 2 (domain B).

The most difficult thing for investigation is region (c), where the nonlinearity of the on-site potential changes sign and is in competition with the nonlinearity of the interparticle interaction. The bifurcation picture for this case is shown in Fig. 1(c). (As usual, the solid curves correspond to the centers and the dashed lines to the saddles.) There are three bifurcation points, but now the frequency $\tilde{\omega}_*$ lies in the gap ($\omega_1<\tilde{\omega}_*<\omega_2$). The corresponding phase portraits for different frequencies are sketched in Fig. 2 (see domain C). (At a certain frequency ω_0 the changeover from one system of levels to the other takes place.) Solitons of C and S type are similar to the solitons of these types in case (a) after changing the values of f and g. Two new types of solitons, of P and R type, are analogous to the solitons of $\tilde{S}$ and $\tilde{C}$ type in region (b), but amplitudes of the f field in the centers of these solitons are smaller than its assimtotes in the infinity. It is of interest to consider the limit of a monoatomic chain for $-1<p<0$. In this limit the gap is absent ($\omega_1=\omega_2=\tilde{\omega}_*$) but nonlinear kinklike solutions lie on each side of the frequency $\omega_1=\sqrt{(2A+\alpha)/m}$. These phase-shift kinks are analogous to phase kinks of the case (a) ($p>1$) in the region $\omega<\omega_1$, and they are similar to the phase kinks of case (b) for $0<p<1$ in the region $\omega>\omega_1$. But now there exists only one type of such solitons for all the frequencies $\omega<\omega_1$ and $\omega>\omega_1$. The phase portraits for different values of the frequency are presented in Figs. 2 (see domain C), and 3(b).

Finally in region (d) with $p<-1$ ($\beta<-8C$), the bifurcation picture is analogous to that in case $p>1$ (a), but the roles of the fields f and g are interchanged and the sequence of bifurcation frequencies changes [compare Figs. 1(a) and 1(d)]. Now the nonlinearity of the on-site potential is very soft, and this fact is more important than the hard nonlinearity of interpretable interaction. The phase portraits for the case $p<-1$ (in Fig. 2, domain D) are similar to those for case $p>1$ (Fig. 2, region A), and in limit $M=m$ we have the same phase-shift kink, but in the area $\omega<\omega_1$.

We supposed in our investigation that the parameters λ and μ in Eqs. (32) and (33) are equal. This is correct in the main approximation for $M-m\ll M$. In the next approximation [for $\lambda/\mu=1+(M-m)/2m$] the boundary between domains (a) and (b) splits into the narrow region with the width $\Delta p\sim(M-m)/m$ [9], but we do not discuss this effect now.

VI. GAP SOLITONS IN MODULATED NONLINEAR OPTICAL MEDIA

For the nonlinear optical model, the functions f and g obey Eqs. (32) and (33) with $\lambda=\mu=1$, $\sigma=-1$, and $p=q=-1$. The bifurcation picture for the dependencies of the singular points g and f on the frequency in this case is shown schematically in Fig. 1(e). It is the same as in Sec. V in the limit $p=1$, but now the bifurcation picture is turned over as the nonlinearity is soft and the values g and f switch roles. The only fixed point for $\omega>\omega_2$ is the center at $g=f=0$, and the soliton solutions are absent. At $\omega=\omega_2$ the first bifurcation occurs: the center is split, and for the frequency range in $\omega_1<\omega<\omega_2$ the system possesses a saddle point at $g=f=0$ and two centers at points $g=0$ and $f=\pm\sqrt{\omega_2-\omega}$. In this frequency gap there exist the two separatrix loops, which correspond to the usual gap solitons [1]. At $\omega=\omega_1$ the second bifurcation occurs: the saddle $g=f=0$ splits into a center at the point ($f=0,g=0$) and two new saddle points ($f=0,g=\pm\sqrt{\omega_1-\omega}$), which are bound together by two different separatrix loops, corresponding to the two different types of soliton solutions [2,3]. In nonlinear modulated optical media the gap solitons have the form of S and C solitons in a diatomic elastic chain, where the values g and f switch roles. But in the optical model the third bifurcation and K-type solitons are absent. Therefore, in the homogeneous limit phase kinks are also absent.

VII. GAP SOLITONS IN A MODULATED FERROMAGNETIC CHAIN

In a diatomic easy-axis ferromagnetic chain with different length of sublattice spins, Eqs. (27) and (28) are reduced to the system of equations (32) and (33), with $\sigma=-1$, $p=-\zeta$, $q=-\zeta(m_0/M_0)^2$, $\lambda=1$, and $\mu=m_0/M_0$, where ζ

$=\beta M_0/2Am_0 \ll 1$. In this example not only are the coefficients in linear terms of Eqs. (32) and (33) for the values g and f different because of the modulation of media, but the coefficients in nonlinear terms are different for $M_0 \neq m_0$ too. But as we used the resonance approximation for the small value $(\omega-\omega_1)/\omega_1$, we must put $M_0 \cong m_0$. The bifurcation picture in this case is shown in Fig. 1(f). For $\omega \geqslant \omega_1$ it is the same as in Sec. VI for optical media, but now there are three bifurcation points. For the frequency range in $\omega_1 < \omega < \omega_2$ the system possesses a saddle point at $g=f=0$, and two centers at points $g=0$ and $f=\pm\sqrt{(\omega_2-\omega)/\zeta}$. Here we have the usual gap solitons. At $\omega=\omega_1$ the saddle $g=f=0$ splits into a center $(f=0, g=0)$ and two new saddle points $[f=0, g=\pm\sqrt{(\omega_1-\omega)/\zeta}]$. But now the third bifurcation take place at $\omega=\tilde{\omega}_*=\omega_1-(\omega_2-\omega_1)\zeta/(1-\zeta)<\omega_1$. At $\omega=\tilde{\omega}_*$ the saddle points $f=0$ and $g=\pm\sqrt{(\omega_1-\omega)/\zeta}$ split into a center at points $f=0$ and $g=\pm\sqrt{(\omega_1-\omega)/\zeta}$ and two saddle points $f_0=\pm\sqrt{(\tilde{\omega}_*-\omega)/(1+\zeta)}$ and $g_0 =\pm\sqrt{f_0^2+(\omega_2-\omega_1)/(1-\zeta)}$ [Fig. 1(f)]. This picture is slightly like Fig. 1(b), but the fields g and f switch roles and the succession of the bifurcations changes direction. The phase portrait and the distribution of the fields g and f are now topologically as in Fig. 2, domain B. Analysis shows that for $\omega<\tilde{\omega}_*$ four different types of separatrices S, C, $\tilde{S}$, and $\tilde{C}$ coexist, none of the field vanishes at infinity, and the principal field difference between S and C solitons disappear. In the limit of monoatomic ferromagnetic with $M_0=m_0$ solitons of C, S, $\tilde{S}$, and $\tilde{C}$ type transform into ''phase kinks'' as in the diatomic elastic chain.

VIII. GAP SOLITONS IN AN EASY-PLANE ANTIFERROMAGNET IN AN EXTERNAL MAGNETIC FIELD

In the case of an easy-plane antiferromagnet in the presence of a magnetic field in the easy plane, the dynamical equations (30) and (31) are reduced to Eqs. (32) and (33) with $\lambda=\mu=1$, $\sigma=-1$, $p=-(1+2\gamma/J-\beta h/2J)/3$, and $q=-(1+2\gamma/J+\beta h/2J)/3$. The frequencies of boundaries of the gap $(\omega_{2,1}\cong 1+2\gamma/J\pm\beta h/2J)$ depend on the magnetic field like parameters q and p, but the dependences $q(h)$ and $p(h)$ are unessential. So in the basic approximation one can consider that $p\cong q\cong -\frac{1}{3}$. This is the same situation as for the nonlinear diatomic chain without on-site potential but with a soft nonlinearity. The bifurcation picture has qualitatively the form represented in Fig. 1(f). But now the third bifurcation frequency is $\tilde{\omega}_*=(3\omega_1-\omega_2)/2$, and the coordinates of the fixed points on the phase portrait are the following: the center at $g=f=0$ for $\omega>\omega_2$; the saddle at $g=f=0$ and the centers at $g=0$ and $f=\pm\sqrt{3(\omega_2-\omega)}$ for $\omega_1<\omega<\omega_2$; the centers at $g=f=0$ and $g=0$; $f=\pm\sqrt{3(\omega_2-\omega)}$, and the saddles at $f=0$ and $g=\pm\sqrt{3(\omega_1-\omega)}$ for $\tilde{\omega}_*<\omega<\omega_1$; the centers at $g=f=0$, $g=0$, $f=\pm\sqrt{3(\omega_2-\omega)}$, $f=0$, and $g=\pm\sqrt{3(\omega_1-\omega)}$; and the saddles at $f=\pm\sqrt{3(\tilde{\omega}_*-\omega)/4}$, $g=\pm\sqrt{3(\omega_2-\omega_1)/2+f^2}$.

IX. CONCLUSION

The main result of this paper is a qualitative analysis of all kinds of small-amplitude excitations with frequencies lying in the gap and near the gap of the spectrum of linear waves for modulated media. Starting with four different nonlinear modulated systems for the small value of the modulation in a long-wave resonant approximation, we obtain a simple system of cubic ordinary nonlinear differential equations of first order for the effective dynamical system with one degree of freedom. The nonlinear part of these differential equations includes two independent parameterss and can be presented in the following forms: pf^3+sfg^2 and pg^3+sgf^2. A full qualitative analysis of the system with different relations between parameters p and s was carried out, and a general classification of all types of gap and near-gap solitons of this system is given in the paper. We investigate transformations of the system phase portrait and the bifurcation picture of soliton solutions with the frequency of the gap soliton and under changes of the system parameters. The results are illustrated by the examples of diatomic elastic crystals, the two-sublattice ferromagnet, and the easy-plane antiferromagnet.

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